HYERS-ULAM STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS

P.PALANI
Assistant Professor
Department of Mathematics
Sri Vidya Mandir Arts & Science College
Uthangarai, Krishnagiri (DT)-636902, T.N. India.

S.JAIKUMAR
Assistant Professor
Department of Mathematics
Sri Vidya Mandir Arts & Science College
Uthangarai, Krishnagiri (DT)-636902, T.N. India.

Abstract
In this paper, we establish the general solution and the generalized Hyers-Ulam stability problem for the equation
\[ f(2x+y) + f(2x-y) = f(x+y) + f(x-y) + 6f(x), \]
(1)

1. Introduction
In 1940, S.M. Ulam [20] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphisms:

It is significant for us to decrease the possible estimator of the stability problem for the functional equations. This work is possible if we consider the stability problem in the of Hyers-Ulam-Rassias for a functional equations(1). As a result, we have much better possible upper bounds for the equations (1) than those of Czerwik [4] and Skof-Cholewa[3].

Solution of \[ f(2x+y) + f(2x-y) = f(x+y) + f(x-y) + 6f(x), \]
Let \( \mathbb{R}^+ \) denote the set of all nonnegative real numbers and let both \( E_1 \) and \( E_2 \) be the vector spaces.

We here present the general solution of (1)

**Theorem 1**
Let \( \phi : \mathbb{R}^2 \to \mathbb{R}^+ \) be a function such that
\[
\sum_{i=0}^{\infty} \frac{\phi(2^i x, 0)}{4^i} = \left( \sum_{i=1}^{\infty} 4^i \phi \left( \frac{x}{2^i}, 0 \right) \right), \text{respectively}
\]

(2)

Converges and

\[
\lim_{n \to \infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0 \quad \left( \lim_{n \to \infty} 4^n \phi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) = 0 \right), \forall x, y \in E_1
\]

(3)

Suppose that a function \( f : X \to Y \) Satisfies

\[
\| f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 6f(x) \| \leq \phi(x, y), \forall x, y \in E_1
\]

(4)

For all \( x, y \in X \). Then there exists a unique quadratic function \( T : X \to Y \) Which Satisfies the equation (2.3) and the inequality

\[
\| f(x) - T(x) \| \leq \frac{1}{8} \sum_{i=0}^{\infty} \phi(2^i x, 0)
\]

(5)

\[
\left( \| f(x) - T(x) \| \leq \frac{1}{8} \sum_{i=1}^{\infty} 4^i \phi \left( \frac{x}{2^i}, 0 \right) \right),
\]

for all \( x \in X \). The function \( T \) is given by

\[
T(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n}
\]

(6)

for all \( x \in X \).

**Proof:**

Putting \( y = 0 \) in \( f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 6f(x) \), and divided by \( 8 \), we have

\[
\| \frac{f(2x)}{4} - f(x) \| \leq \frac{1}{8} \phi(x, 0)
\]

(7)

for all \( x \in X \). Replacing \( x \) by \( 2x \) in (7) and dividing by \( 4 \) and summing the resulting inequality with (7), we get

\[
\| \frac{f(2^2 x)}{4^2} - f(x) \| \leq \frac{1}{8} \left[ \phi(x, 0) + \frac{\phi(2x, 0)}{4} \right]
\]

(8)

for all \( x \in X \). Using the induction on a positive integer \( n \), we obtain that

\[
\| \frac{f(2^n x)}{4^n} - f(x) \| \leq \frac{1}{8} \sum_{i=0}^{n-1} \phi(2^i x, 0)
\]

(9)

\[
\leq \frac{1}{8} \sum_{i=0}^{\infty} \phi(2^i x, 0)
\]

for all \( x \in X \). In order to prove convergence of the sequence \( \left\{ \frac{f(2^n x)}{4^n} \right\} \), we divide inequality (9)

by \( 4^m \) and also replace \( x \) by \( 2^m x \) to find that for \( n, m > 0 \),

---

20
\[ \left\| \frac{f(2^n 2^m x) - f(2^m x)}{4^n} \right\| \leq \frac{1}{8} \sum_{i=0}^{n-1} \phi(2^i 2^m x, 0) \] 
\[ \leq \frac{1}{8} \sum_{i=0}^{n-1} \phi(2^i 2^m x, 0) \]

Since the right hand side of the inequality tends to 0 as \( m \) tends to infinity, the sequence \( \left\{ \frac{f(2^n x)}{4^n} \right\} \) is a Cauchy sequence. Therefore, we may define \( T(x) = \lim_{n \to \infty} 2^{-2n} f(2^n x) \) for all \( x \in X \).

By letting \( n \to \infty \) in (9), we arrive at the formula (5).

To show that \( T \) satisfies the equation (2.3), replace \( x,y \) by \( 2^n x, 2^n y \), respectively in
\[ f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 6f(x) \]
and divided by \( 4^n \), then it follows that
\[ 4^n \left\| f(2^n (2x + y)) + f(2^n (2x - y)) - f(2^n (x + y)) - f(2^n (x - y)) - 6f(2^n x) \right\| \leq 4^{-n} \phi(2^n x, 2^n y). \]

Taking the limits as \( n \to \infty \), we find that \( T \) satisfies (2.3) for all \( x,y \in X \).

To prove the uniqueness of the quadratic function \( T \) subject to (1), let us assume that there exists a quadratic function \( S: X \to Y \) which satisfies (2.3) and the inequality (1).

Obviously, we have \( S(2^n x) = 4^n S(x) \) and \( T(2^n x) = 4^n T(x) \) for all \( x \in X \) and \( n \in \mathbb{N} \). Hence it follows from (1) that
\[ \left\| S(x) - T(x) \right\| = 4^{-n} \left\| S(2^n x) - T(2^n x) \right\| \]
\[ \leq 4^{-n} \left( \left\| S(2^n x) - f(2^n x) \right\| + \left\| f(2^n x) - T(2^n x) \right\| \right) \]
\[ \leq \frac{1}{4^n} \sum_{i=0}^{n-1} \phi(2^i 2^n x, 0) \]

For all \( x \in X \). By letting \( n \to \infty \) in the preceding inequality, we immediately find the uniqueness of \( T \). This completes the proof of the theorem.

Throughout this paper, let \( B \) be a unital Banach algebra with norm \( \| \cdot \| \), and let \( B_1 \) and \( B_2 \) be the left Banach \( B \)-modules with norm \( \| \cdot \| \) and \( \| \cdot \| \), respectively.

A quadratic mapping \( Q: B_1 \to B_2 \) is called \( B \)-quadratic if
\[ Q(ax) = a^2 Q(x), \quad \forall a \in B, \forall x \in B_1. \]

**Corollary 1.1.**

Let \( \phi: B_1 \times B_1 \to \mathbb{R}^+ \) be a function satisfies (1) and (2) for all \( x,y \in B_1 \). Suppose that a mapping \( f: B_1 \to B_2 \) satisfies
\[ \left\| f(2ax + ay) + f(2ax - ay) - \alpha^2 f(x + y) - \alpha^2 f(x - y) - 6\alpha^2 f(x) \right\| \leq \phi(x,y) \]

For all \( \alpha \in B(\| \alpha \| = 1) \) and for all \( x,y \in B_1 \) and \( f \) is measurable or \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in B_1 \). Then there exists a unique \( B \)-quadratic mapping \( T: B_1 \to B_2 \), defined by (5), which satisfies the equation (2.3) and the inequality (1) for all \( x \in B_1 \).
Proof:
By theorem 3.1, it follows from the inequality of the statement for $\alpha = 1$ that there exists a unique quadratic mapping $T : b B_i \rightarrow b B_2$ satisfying the inequality (3.4) for all $x \in b B_i$. Under the assumption that $f$ is measurable or $f(tx)$ is continuous in $x \in \mathbb{R}$ for each fixed $x \in b B_i$, by the same reasoning as the proof of [5], the quadratic mapping $T : b B_i \rightarrow b B_2$ satisfies

$T(tx) = t^2 T(x)$, $\forall x \in b B_i \forall t \in \mathbb{R}$.

That is, $T$ is $B$-quadratic. For each fixed $\alpha \in B(\alpha \neq 0)$, replacing $f$ by $T$ and setting $y = 0$ in (2.3), we have $T(\alpha x) = \alpha^2 T(x)$ for all $x \in b B_i$. The last relation is also true for $\alpha = 0$. For each element $\alpha \in B(\alpha \neq 0), a = |a| \cdot \frac{\alpha}{|a|}$.

Since $T$ is $B$-quadratic and $T(\alpha x) = \alpha^2 T(x)$ for each element $\alpha \in B(\alpha \neq 1)$,

$T(ax) = T \left( |a| \cdot \frac{a}{|a|} \cdot x \right)$

$= |a|^2 \cdot T \left( \frac{\alpha}{|a|} \cdot x \right)$

$= |a|^2 \cdot \frac{\alpha^2}{|a|^2} T(x)$

$= a^2 T(x)$, $\forall \alpha \in B(\alpha \neq 0), \forall x \in b B_i$.

So the unique $B$-quadratic mapping $T : b B_i \rightarrow b B_2$, is also $B$-quadratic, as desired.

This completes the proof of the corollary.

Corollary 1.2.
Let $E_1$ and $E_2$ be Banach spaces over the complex field $\mathbb{C}$, and let $e \geq 0$ be a real number.

Suppose that a mapping $f : E_1 \rightarrow E_2$ satisfies

$\|f(2ax + \alpha y) + f(2ax - \alpha y) - \alpha^2 f(x + y) - \alpha^2 f(x - y) - 6\alpha^2 f(x)\| \leq \varepsilon$

for all $\alpha \in \mathbb{C}(\alpha \neq 1)$ and for all $x, y \in E_1$ and $f$ is measurable or $f(tx)$ continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Then there exists a unique $B$-quadratic mapping $T : E_1 \rightarrow E_2$ which satisfies the equation (1.3) and the inequality

$\|f(x) - T(x)\| \leq \frac{\varepsilon}{6}, \forall x \in E_1$.

Corollary 1.3.
Let $X$ and $Y$ be a real normed space and Banach space respectively, and let $e, p, q$ be real numbers such that $e \geq 0, q > 0$ and either $p, q < 2$ or $p, q > 2$. Suppose that a function $f : X \rightarrow Y$ satisfies

$\|f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 6f(x)\| \leq e \left( \|x\|^p + \|y\|^q \right)$

for all $x, y \in X$. Then there exists a unique quadratic function $T : X \rightarrow Y$ which satisfies the equation (1.3) and the inequality
\[ \|f(x) - T(x)\| \leq \frac{\varepsilon}{2|4 - 2^p|}\|x\| \]

for all \( x \in X \) and for all \( x \in X - \{0\} \) if \( p < 0 \).

The function \( T \) is given by \( T(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n} \) if \( p,q < 2 \)

\[ T(x) = \lim_{n \to \infty} 4^n f \left( \frac{x}{2^n} \right) \text{ if } q > 2 \]

for all \( x \in X \). Further, if for each fixed \( x \in X \) the mapping \( t \to f(tx) \) from \( X \) to \( Y \) is continuous, then \( T(rx) = r^2T(x) \) for all \( r \in X \).

The proof of the corollary.

**Corollary 1.4**

Let \( X \) and \( Y \) be a real normed space and a Banach space, respectively, and let \( \varepsilon \geq 0 \) be real number. Suppose that a function \( f : X \to Y \) satisfies

\[ \|f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 6f(x)\| \leq \varepsilon \]

for all \( x, y \in X \). Then there exists a unique quadratic function \( T : X \to Y \) defined by

\[ T(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n} \]

which satisfies the equation (1.3) and the inequality

\[ \|f(x) - T(x)\| \leq \frac{\varepsilon}{6} \] (12)

for all \( x \in X \). Further, if for each fixed \( x \in X \) the mapping \( t \to f(tx) \) from \( X \) to \( Y \) is continuous, then \( T(rx) = r^2T(x) \) for all \( r \in X \).

**Corollary 1.5**

Let \( X \) and \( Y \) be a real normed space and Banach space, respectively, and let \( \varepsilon \geq 0, 0 < p \neq 2 \) be real number. Suppose that a function \( f : X \to Y \) satisfies

\[ \|f(2x + y) + f(2x - y) - f(x + y) - f(x - y) - 6f(x)\| \leq \varepsilon (\|x\|^p + \|y\|^p) \]

for all \( x, y \in X \). Then there exists a unique quadratic function \( T : X \to Y \) which satisfies the equation (1.3) and the inequality

\[ \|f(x) - T(x)\| \leq \frac{\varepsilon}{2 |9 - 3^p|}\|x\|^p \]

for all \( x \in X \). The function \( T \) is given by

\[ T(x) = \lim_{n \to \infty} \frac{f(3^n x)}{9^n} \text{ if } 0 < p < 2 \]

\[ T(x) = \lim_{n \to \infty} 9^n f \left( \frac{x}{3^n} \right) \text{ if } q > 2 \]

for all \( x \in X \). Further, if for each fixed \( x \in X \) the mapping \( t \to f(tx) \) from \( X \) to \( Y \) is continuous, then \( T(rx) = r^2T(x) \) for all \( r \in X \).

**REFERENCES**


