MEYNIEL’S GRAPHS ARE STRONGLY PERFECT

S.JAIKUMAR
Assistant Professor
Department of Mathematics
Sri Vidya Mandir Arts & Science College
Uthangarai, Krishnagiri (DT)-636902, T.N. India

P.PALANI
Assistant Professor
Department of Mathematics
Sri Vidya Mandir Arts & Science College
Uthangarai, Krishnagiri (DT)-636902, T.N. India

ABSTRACT

The graph-theoretic notions used here are those of [3]. A graph is called perfect if the chromatic number of each of its induced subgraph H equals the number of vertices in the largest clique of H; it is called strong perfect [4] if each of its induced subgraphs H contains a stable set which meets all the maximal cliques in H. (Here, as usual, ‘maximal’ is meant with respect to set-inclusion.) It is easy to show that every strongly perfect graph is perfect. Meyniel[5] proved that a graph is perfect whenever each of its odd cycles of length at least five has at least two chords. The purpose of this paper is to strengthen Meynél’s result as follows:

Theorem:

If every odd cycle of length at least five in a graph G has at least two chords then G is strongly perfect.

The following useful observation has been made by Meynél.

Lemma:

If a graph G = (V, E) contains an odd cycle [x₀, x₁, x₂, ..., x₂t, x₀] such that the path [x₁, x₂, ..., x₂t] is chordless and x₀ is non-adjacent to at least one xₖ, then G contains an odd cycle of length at least five with at most one chord.
Proof:

If $x_0 x_2 \in E$ then consider the largest $i$ such that $x_0$ is adjacent to $x_1, x_2, \ldots, x_i$ and the smallest $j$ such that $j > i$ and $x_0 x_j \in E$: the cycle $[x_0, x_i, \ldots, x_j, x_0]$ has no chords, the cycle $[x_0, x_{i-1}, x_i, \ldots, x_j]$ has precisely one chord, and one of these two cycles is odd. If $x_0 x_2 \not\in E$ then consider the smallest even $j$ such that there exists $x_0 x_j \in E$ and the largest $i$ such that $i \leq j-2$ and $x_0 x_i \in E$: the odd cycle $[x_0, x_i, \ldots, x_j, x_0]$ has at most one chord.

By a starter in a graph $G$, we shall mean a cycle $w v_0 v_1 \ldots v_k w$ such that

1. $v_0$ is adjacent to none of the vertices $v_2, v_3, \ldots, v_k$.
2. $w$ is not adjacent to $v_1$.
3. Some stable set $S$, containing $v_1$ and $v_k$, meets all the maximal cliques in $G - v_0$.

Lemma:

If a graph contains a starter then it contains an odd cycle of length at least five with at most one chord.

Proof:

We shall present an informal description of an efficient algorithm which, given any starter in $G = (V, E)$, finds the desired cycle. No generality is lost by assuming that

1. $[v_1, v_2, \ldots, v_k, w]$ is the shortest path from $v_1$ to $w$ with the next-to-last vertex in $S$ and all the vertices except the first and the last non-adjacent to $v_0$.

In particular, (iv) implies that

2. The path $[v_1, v_2, \ldots, v_k]$ has no chords.

Next, we may assume that

3. Every $v_i$ adjacent to $w$ has an even subscript $r$;

Otherwise the desired cycle can be found at once by applying lemma to the odd cycle $w v_0 \ldots v_{i-1} v_i v_{i+1} \ldots v_k w$.

Write $y \in S^*$ if $y \in S$ and $y$ is adjacent to two consecutive vertices $v_j, v_{j+1}$ on the path $[v_0, v_1, v_2, \ldots, v_k]$. We may assume that

4. No $y \in S^*$ is adjacent to $v_0$;

Otherwise the desired cycle can be found at once by applying lemma to any odd cycle $[y v_0 \ldots y v_i y]$. Now it follows that

5. No $y \in S^*$ is adjacent to $w$;

Otherwise (iv) would be contradicted by $v_0 v_1 \ldots v_i y w$ such that $i$ is the smallest subscript with $y v_i \not\in E$. Next, we may assume that

6. Each $y \in S^*$ is adjacent to at least three vertices on the path $v_0 v_1 \ldots v_k$;

Otherwise the desired cycle is $[w, v_1, \ldots, v_j, y, v_{j+1}, \ldots, v_s, w]$ with $r$ standing for the largest subscript such that $r \leq j$, $w v_r \in E$ and $s$ standing for the smallest subscript such that $s \geq j+1$, $w v_s \in E$. Now it follows that

7. Each $y \in S^*$ is adjacent to precisely three vertices $v_{i-1}, v_i, v_{i+1}$ on the path $[v_0, v_1, \ldots, v_k]$;

Otherwise (iv) would be contradicted by $v_1 \ldots v_i y v_s \ldots v_k$ such that $r$ is the smallest subscript with $y v_r \in E$ and $s$ is the largest subscript with $y v_s \in E$.

Now observe that, for any $y \in S^*$ adjacent to $v_i, v_1$, and $v_{i+1}$, the substitution of $y$ for $v_i$ in the original starter yields a new starter with a smaller $S^*$. Repeating this operation as many times as possible, we eventually obtain a starter $w v_0 v_1 \ldots v_k w$ satisfying (iv) and having an empty $S^*$.
References
2) C. Berge, Sur une conjecture relative au problem des codes optimaux, comm. 13eme assemblee generale de l’URSI, Tokyo,1962
3) C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973)
4) C. Berge and P. Duchet , strongly perfect graphs(this volume, pp. 57-61)