GENERALIZATION OF THE HYERS-ULAM – RASSIAS STABILITY OF APPROXIMATELY ADDITIVE MAPPINGS

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Abstract
In this paper we prove a generalization of the stability of approximately additive mappings in the spirit of Hyers, Ulam, and Rassias.

1. Introduction
Questions concerning the stability of functional equations seem to have been first studied by Ulam[6]. In 1941 Hyers[1] showed that if \( f : E_1 \to E_2 \), with \( E_1 \) and \( E_2 \) Banach spaces, such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \delta, \quad \text{for all } x, y \in E_1,
\]

Then there exist a unique \( T : E_1 \to E_2 \) such that

\[
T(x + y) = T(x) + T(y)
\]

And

\[
\|f(x) - T(x)\| \leq \delta, \quad x, y \in E_1, \text{ and if } f(tx) \text{ is continuous in } t \text{ for each fixed } x, \text{ then } T \text{ is a linear mapping.}
\]

In 1978 a generalized solution to the Ulam problem for approximately linear mapping was given by Rassias[5]:

Consider \( E_1, E_2 \) to be two Banach spaces and \( f : E_1 \to E_2 \), to be mapping such that \( f(tx) \) is continuous in \( t \) for each fixed \( x \). Assume that there exist \( \theta \geq 0 \) and \( p \in [0,1] \) such that

\[
f(tx) = tf(x), \quad x \in E_1, \quad t \in [0,1],
\]
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\[ \|f(x + y) - f(x) - f(y)\| \leq \theta, \text{ for all } x, y \in E_1. \]

Then there exists a unique linear mapping \( T : E_1 \rightarrow E_2 \) such that

\[ \frac{\|f(x) - T(x)\|}{\|x\|} \leq \frac{2\theta}{2 - 2^p}, \quad x, \in E_1. \]


We denote by \((G, +)\) an abelian group, by \((X, (, . )x)\) a Banach space, and by \( \varphi : G \times G \rightarrow [0, \infty) \) a mapping such that

\[ \varphi(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty \quad (1) \]

\( x, y \in G. \)

**Theorem.** Let \( f : G \rightarrow X \) be such that

\[ \|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y) \text{ for all } x, y \in G. \quad (2) \]

Then there exists a unique mapping \( T : g \rightarrow X \) such that

\[ T(x + y) = T(x) + T(y), \text{ for all } x, y \in G \quad (3) \]

And

\[ \|f(x) - T(x)\| \leq \frac{1}{2} \varphi(x, x), \quad x \in G. \quad (4) \]

**Proof.**

For \( x = y \) inequality (2) implies

\[ \|f(x) - 2f(x)\| \leq \varphi(x, x). \]

Thus

\[ \|2^{-1} f(2x) - f(x)\| \leq \varphi(x, x), \quad x \in G. \quad (5) \]

Replacing \( x \) by 2\( x \), inequality (5) gives

\[ \|2^{-1} f(2^2 x) - f(2x)\| \leq \frac{1}{2} \varphi(2x, 2x), \quad x \in G. \quad (6) \]

From (5) and (6) it follows that

\[ \|2^{-2} f(2^2 x) - f(x)\| \leq \|2^{-2} f(2^2 x) - 2^{-1} f(2x)\| + \|2^{-1} f(2x) - f(x)\|. \]
\[ \leq 2^{-1} \frac{1}{2 \varphi(2^2,2^2)} + 1 \frac{1}{2 \varphi(x,x)}. \]

Hence \[ \|2^{-2}f(2^2x) - f(x)\| \leq 1/2[\varphi(x,x) + 1/2 \varphi(2^2,2^2)] \] for all \( x \in G \). \hspace{1cm} (7)

Replacing \( x \) by \( 2x \), inequality (7) becomes

\[ \|2^{-3}f(2^3x) - f(x)\| \leq 1/2[\varphi(2x,2x) + 1/2 \varphi(2^2,2^2)]. \]

Thus

\[ \|2^{-3}f(2^3x) - f(x)\| \leq 1/2[\varphi(x,x) + 1/2 \varphi(2x,2x) + 1/2 \varphi(2^2,2^2)] \] \hspace{1cm} (8)

for all \( x \in G \).

Applying an induction argument to \( n \) we obtain

\[ \|2^{-n}f(2^nx) - f(x)\| \leq 1/2 \sum_{k=0}^{n-1} 2^{-k} \varphi(2^k x,2^k x) \] for all \( x \in G \). \hspace{1cm} (9)

Indeed,

\[ \|2^{-(n+1)}f(2^{n+1}x) - f(x)\| \leq \|2^{-(n+1)}f(2^{n+1}x) - 2^{-1} f(2^x)\| + \|2^{-1} f(2^x) - f(x)\|. \]

And with (9) and (5) we obtain

\[ \|2^{-(n+1)}f(2^{n+1}x) - f(2^x)\| \leq 2^{-1} 1 \frac{1}{2k} = \sum_{k=0}^{n-1} 2^{-k} \varphi(2^{k+1} x,2^{k+1} x) + 1/2 \varphi(x,x) \]

\[ = 1/2 \sum_{k=0}^{n} 2^{-k} \varphi(2^k x,2^k x). \]

We claim that the sequence \( \{2^{-n}f(2^n x)\} \) is a Cauchy sequence. Indeed, for \( n > m \) we have

\[ \|2^{-n}f(2^n x) - 2^{-m}f(2^m x)\| = 2^{-m} \|2^{-(n-m)}f(2^{n-m-2^m} x) - f(2^m x)\| \]

\[ \leq 2^{-m} 1 \frac{1}{2k} \sum_{k=0}^{n-m-1} 2^{-k} \varphi(2^{k+m} x,2^{k+m} x) \]

\[ = 1/2 \sum_{p=m}^{n-1} 2^{-p} \varphi(2^p x,2^p x). \]

Taking the limit as \( m \to \infty \) we obtain
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\[
\lim_{n \to \infty} 2^{-n} f(2^n x) = 0.
\]

Because of the fact that X is a Banach space it follows that the sequence \( \{2^{-n} f(2^n x)\} \) converges.

Denote

\[
T(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}.
\]

We claim that T satisfies (3).

From (2) we have

\[
\|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\| = \phi(2^n x, 2^n y)
\]

for all \( x, y \in G \). Therefore

\[
2^{-n} f(2^n x + 2^n y) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y) \leq 2^{-n} \phi(2^n x, 2^n y).
\]

(10)

From (1) it follows that

\[
\lim_{n \to \infty} 2^{-n} \phi(2^n x, 2^n y) = 0.
\]

Then (10) implies

\[
\|T(x + y) - T(x) - T(y)\| = 0.
\]

To prove (4), taking the limit in (9) as \( n \to \infty \), we obtain

\[
\|T(x) - f(x)\| \leq 1/2 \phi(x, x), \quad \text{for all } x \in G.
\]

It remains to show that T is uniquely defined. Let \( f : G \to X \) be another such mapping with \( F(x + y) = F(x) + F(y) \).

And (4) satisfied.

Then

\[
\|T(x) - F(x)\| = \|2^{-n} T(2^n x) - 2^{-n} F(2^n x)\|
\]

\[
\leq \|2^{-n} T(2^n x) - 2^{-n} f(2^n x)\| + \|2^{-n} f(2^n x) - 2^{-n} F(2^n x)\|
\]

\[
= 2^{-n} \phi(2^n x, 2^n x) + 2^{-n} \phi(2^n x, 2^n x)
\]

\[
= 2^{-n} \phi(2^n x, 2^n x)
\]
\[
= 2^{-n} \sum_{k=0}^{\infty} 2^{-k} \phi(2^{k+n} x, 2^{k+n} x)
= \sum_{p=n}^{\infty} 2^{-p} \phi(2^p x, 2^p x).
\]

Thus
\[
\|T(x) - F(x)\| \leq \sum_{p=n}^{\infty} 2^{-p} \phi(2^p x, 2^p x). \quad \text{for all } x \in G \tag{11}
\]

Taking the limit in (11) as \( n \to \infty \) we obtain
\[
T(x) = F(x) \text{ for all } x \in G.
\]

**APPLICATION**

Let \( G \) be a normed linear space and define \( H : R_+ \times R_+ \to R_+ \) and \( \varphi_0 : R_+ \to R_+ \) such that

\[
\varphi_0(\lambda) > 0 \quad \text{for all } \lambda > 0,
\]

\[
\varphi_0(2) < 2
\]

\[
\varphi_0(2\lambda) \leq \varphi_0(2)\varphi_0(\lambda), \quad \text{for all } \lambda > 0
\]

\[
H(\lambda t, \lambda s) \leq \varphi_0(\lambda)H(t, s) \quad \text{for all } t, s \in R_+, \lambda > 0.
\]

We take in our theorem

\[
\varphi(x, y) = H(\|x\|, \|y\|).
\]

Then

\[
\varphi(2^k x, 2^k y) = H(2^k \|x\|, 2^k \|y\|)
\leq \varphi_0(2^k)H(\|x\|, \|y\|)
\leq (\varphi_0(2^k))^k H(\|x\|, \|y\|)
\]

And because \( \varphi_0(2) < 2 \) we have

\[
\varphi(x, y) \leq \sum_{k=0}^{\infty} 2^{-k} (\varphi_0(2))^k H(\|x\|, \|y\|)
\]
\[
= \frac{1}{1-(\varphi_0(2)/2))} H(\|x\|,\|y\|),
\]

And the relation (4) becomes
\[
\|f(x)-T(x)\| \leq \frac{1}{2} \varphi(x,x) \leq \frac{1}{2-\varphi_0(2)} H(\|x\|,\|y\|)
\]

Or
\[
\|f(x)-T(x)\| \leq \frac{1}{2-\varphi_0(2)} \varphi_0(x) H(1,1).
\]

Remark: The above result generalizes results of Isac and Rassias [3,4] and Rassias [5] because if \(f(tx)\) is continuous in \(t\) for each fixed \(x\) and
\[
T(x) = \lim_{n \to \infty} 2^{-n} f(2^n x),
\]

Then \(T\) is a linear mapping (see [5]).

REFERENCES


